if and only if the following equation is satisfied by the coefficients:

$$[\int_s \bar{w}_i(x, y)w_j(x, y)dxdy]_{n\times n} \{b_i\}_{n\times 1} = \{\int_s \bar{w}_i (x, y)w(x, y)dxdy\}_{n\times 1} \quad (6)$$

provided that the coefficient matrix is nonsingular. The bars denote complex conjugation. It is also shown[1] that the coefficient matrix is always positive semidefinite and is positive definite and nonsingular if and only if the downwash functions due to assumed pressure modes are linearly independent over the surface.

Equation (6) is not yet in a form suitable for digital calculations. Because the integrals are quite complicated, numerical procedures are necessary. Using numerical integration for the coefficient matrix in Eq. (6), it follows that

$$[\bar{w}_i(x_j, y_j)]_{n\times m}\lceil W_i \rfloor_{m\times m}[w_j(x_i, y_i)]_{m\times n}\{b_i\}_{n\times 1} = [\bar{w}_i(x_j, y_j)]_{n\times m}\lceil W_i \rfloor_{m\times m}\{w(x_i, y_i)\}_{m\times 1} \quad (7)$$

where $(x_1, y_1), \ldots, (x_m, y_m)$ are m integration points and $\lceil W_i \rfloor$ is the diagonal matrix of integration weighting factors. One can see from Eq. (7) that the number of integration points must equal or exceed the number of assumed modes because the rank of the triple product cannot exceed the minimum of the ranks of its factors. Equation (7) can be stated in terms of more familiar matrices by observing that

$$D_{ij} = W_j(x_i, y_i) \quad (8)$$

Therefore, the final form of the least-squares equation is

$$[\bar{D}_{ij}]^T_{n\times m}\lceil W_i \rfloor_{m\times m}[D_{ij}]_{m\times n}\{b_i\}_{n\times 1} = [\bar{D}_{ij}]^T_{n\times m}\lceil W_i \rfloor_{m\times m}\{W(x_i, y_i)\}_{m\times 1} \qquad m \geqslant n \quad (9)$$

If $n = m$ and $[D_{ij}]$ is nonsingular, Eq. (9) for least-square error reduces to Eq. (4) for the collocation solution. The largest portion of computing time in existing programs is devoted to computing $[D_{ij}]$. Therefore, additional time required to perform the indicated matrix multiplications would be negligible. This leads to an important conclusion: existing collocation-based programs can be readily converted to solutions possessing minimum square error by increasing the number of points and replacing the coefficient matrix and right-hand side of Eq. (4) by their triple-product counterparts in Eq. (9). This includes the familiar procedures for real equations as a special case. It is seen that the conversion procedure is identical for mathematically similar problems involving complex valued equations.

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A Passive System for Determining the Attitude of a Satellite

HAROLD D. BLACK*

The Johns Hopkins University, Silver Spring, Md.

I. Introduction

A GROUP at the Applied Physics Laboratory has, during the past several years, developed "solar attitude" detectors.[1] In principle, these detectors are single solar cells placed on orthogonal axes of the satellite. From processing the telemetered voltage output from these cells, we can obtain the direction cosines of the satellite-sun vector.

Relative to the satellite-fixed axes defined by the solar detectors, we can place another set of sensors to measure the direction cosines of another vector in body-fixed coordinates. We have used three orthogonally placed fluxgate magnetometers to measure the direction cosines of the earth's magnetic field. With the direction cosines of two vectors and independent knowledge of the satellite position, then, we derive the attitude matrix of the satellite.

II. Analysis

We will define (in any convenient way) a set of orthogonal axes fixed in the satellite. Relative to this axis, we will then measure, at any particular instant, the required direction cosines.

Associated with the body-fixed axis is a "reference" coordinate system (e.g., inertial coordinates) such that if the relationship of the body axis to the reference system is available, we can then say that the attitude of the satellite is known. To make this statement explicit, we are given the components (r_1, r_2, r_3) of a vector of unit magnitude in body-fixed coordinates. This vector has a representation in another orthogonal coordinate system:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \{A\}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad (1)$$

wherein $\{A\}$ is a (orthogonal) rotational matrix that transforms from body-fixed to, say, inertial coordinates. If $\{A\}$ can be obtained explicitly, then the attitude is known. In the system implemented by us, $\mathbf{r}$ is obtained from the solar attitude detectors, whereas $\mathbf{R}$ is computed from knowledge of the satellite orbit and the position of the sun.

It is intuitively clear that Eq. (1) is sufficient to determine the orientation of the satellite axes relative to the satellite sunline, excepting the degree of freedom involving rotation about this line. To determine all three degrees of freedom, it is sufficient to treat similarly any other linearly independent vector that can be resolved in satellite coordinates. For any vector $\mathbf{b}$, linearly independent from $\mathbf{r}$, we write

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \{A\}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (2)$$

In the system implemented by us, we compute $\mathbf{B}$ from the known position of the satellite and a mathematical model of the earth's magnetic field.[2] $\mathbf{b}$ is obtained from the fluxgate magnetometer telemetry results.

Equations (1) and (2) are then considered as a pair of simultaneous equations to be solved for the attitude matrix $\{A\}$. This solution is quite simple to achieve: Since $\{A\}$ transforms any vector in satellite coordinates, and since $\mathbf{r}$

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*Project Supervisor, Computer Systems, Space Division, Applied Physics Laboratory. Associate Member AIAA.

Table 1 Angle included between satellite polar axis and the local vertical: Satellite 1963 22A, June 22, 1963

Universal time			Along orbit, deg	Across orbit, deg
hr	min	sec		
11	58	53	7.0	−19.1
	59	15	6.3	−18.7
12	00	21	2.5	−16.9
	00	43	4.6	−17.5
	01	49	2.1	−15.6
	02	11	1.5	−14.5
	03	39	2.4	−13.0
	04	01	1.1	−11.9
	07	20	−1.5	− 5.6
	07	42	−1.7	− 5.4
	08	48	−2.2	− 2.9
	09	10	−2.5	− 1.9

and **b** are linearly independent, we obviously have the relationship

$$\begin{pmatrix}(\mathbf{R}\times\mathbf{B})_1\\(\mathbf{R}\times\mathbf{B})_2\\(\mathbf{R}\times\mathbf{B})_3\end{pmatrix} = \{A\}\begin{pmatrix}(\mathbf{r}\times\mathbf{b})_1\\(\mathbf{r}\times\mathbf{b})_2\\(\mathbf{r}\times\mathbf{b})_3\end{pmatrix} \qquad (3)$$

Equations (1–3) are then combined into a single equation:

$$\begin{Bmatrix}R_1 & B_1 & (\mathbf{R}\times\mathbf{B})_1\\R_2 & B_2 & (\mathbf{R}\times\mathbf{B})_2\\R_3 & B_3 & (\mathbf{R}\times\mathbf{B})_3\end{Bmatrix} = \{A\}\begin{Bmatrix}r_1 & b_1 & (\mathbf{r}\times\mathbf{b})_1\\r_2 & b_2 & (\mathbf{r}\times\mathbf{b})_2\\r_3 & b_3 & (\mathbf{r}\times\mathbf{b})_3\end{Bmatrix} \qquad (4)$$

As the matrix on the extreme right of Eq. (4) has columns that are the components of linearly independent vectors, its inverse necessarily exists. Thus, by inverting this matrix, we can solve for the attitude matrix, $\{A\}$.† A simple extension facilitates this inversion: If we use for the third column (rather than the components of $\mathbf{r}\times\mathbf{b}$) the components of $\mathbf{r}\times\mathbf{b}/|\mathbf{r}\times\mathbf{b}|$, and for the first column the components of $\mathbf{b}\times(\mathbf{r}\times\mathbf{b}/|\mathbf{r}\times\mathbf{b}|)$, then the matrix we have to invert is an orthogonal matrix that can be inverted at sight.‡ These details, however, are only of computational interest, as Eq. (4) embodies the crucial idea necessary to obtain the solution.

Once the attitude matrix is obtained, it is a trivial matter to obtain the orientation of the satellite relative to other known coordinate systems. A particularly convenient one that we have used is a coordinate system based on the position and angular momentum vectors of the satellite. Using this coordinate system, we compute the angle that the satellite polar axis forms with the position vector to the center of the earth. Subsequently, we resolve this angle into components along and orthogonal to the orbit plane.

Seemingly convenient for dynamic analyses would be a formal transformation of the $\{A\}$ matrix to Eulerian angles. On the other hand, the work of Marguiles and Goodman[3] has given us "pause for concern."

III. Some Results

The technique described previously was successfully used aboard the 1963 22A satellite. This satellite contained a pioneering experiment to evaluate gravity-gradient stabilization.[4] Sample results obtained are contained in Table 1. These results confirmed that the gravity stabilization system was successful.

IV. Discussion

It is clear that measurements of any two linearly independent vectors in satellite coordinates are sufficient to determine the orientation if these vectors have a known resolution in the reference coordinate system. Usually, the necessary conditions, because of available constraints, are less stringent. In our system, we chose to measure all six quantities and use three available constraints, e.g.,

$$\sum_{i=1}^{3} r_i^2 = \sum_{i=1}^{3} b_i^2 = 1 \qquad \mathbf{r}\cdot\mathbf{b} = \mathbf{R}\cdot\mathbf{B}$$

to minimize the effects of noise in the data. A discussion of this noise analysis would take us outside the scope of this paper.

† The writer is indebted to his colleague, Stuart Haywood of APL/JHU for pointing out this method of solving Eqs. (1) and (2).

‡ The left side of Eq. (4) is, of course, replaced with the corresponding quantities in **R** and B.

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Buckling of Circular Cylindrical Shells in Axial Compression

R. C. Tennyson*
University of Toronto, Toronto, Ontario, Canada

Nomenclature

X, Y	=	$mx/L, ny/2\pi R$
l_x	=	L/m
m, n	=	number of half-waves in the axial and circumferential directions, respectively
x, y, z	=	reference set of orthogonal coordinate axes
u, v, w	=	displacements in the x, y, and z directions, respectively
L	=	shell length
R	=	initial shell radius
t	=	shell wall thickness
A, B, C	=	displacement amplitudes
σ	=	normal stress
τ	=	shear stress
ϵ	=	normal strain
γ	=	shear strain
θ	=	angle measured from x axis counterclockwise

RECENT tests on five circular cylindrical shells having clamped ends subjected to axial compression have yielded buckling loads within 10% of the classical predicted values.[1] All shells were made from a photoelastic plastic and behaved completely elastically, thus permitting repeatable tests. During the course of photoelastically analyzing the shells,[2] high-speed photographs of the buckling process taken through a plane reflection-type polariscope were obtained with a Fastax camera. The change in isoclinic patterns with change in buckle waveshape was recorded as a function of time. Of particular interest is the change in the 45° isoclinic during buckling for a shell with a reflective surface at some intermediate position in the shell's wall (Fig. 1). The following note describes a brief analysis of the initial stages of buckling and compares the classical waveshape with that observed in frames 3 to 6 of Fig. 1.

From the theory of photoelasticity,[3] it is known that an isoclinic of parameter θ defines the locus of points in a stressed

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* Research Assistant, Institute for Aerospace Studies. Student Member AIAA.